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MULTI-GROUP, MULTI-REFLECTOR PILE THEORY

By
H. L. Garabedian
A. S. Householder

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Clinton Laboratories
Oak Ridge, Tennessee

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Section VI

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G. Young, Section Chief
CLINTON LABORATORIES
Oak Ridge, Tennessee

Multi-Group, Multi-Reflector Pile Theory

1. Introduction. The differential equations involved in the multi-group, multi-reflector pile theory studied in this paper are homogeneous partial differential equations of the second order. Associated with the pile or any of the reflectors is a system of as many equations as there are groups. Thus, if there are n groups and ν reflectors there is a system of n second order equations in each of the $\nu + 1$ regions, the solutions of the $\nu + 1$ systems satisfying appropriate continuity conditions at the boundaries.

There are three types of piles considered: the infinite slab, the infinite cylinder, and the sphere. Moreover, symmetrical characteristics of the problems are such that the partial differential equations reduce to ordinary differential equations. Although the equations are linear, the coefficients are constant only in the case of the slab pile; otherwise, they are functions of the independent variable.

Ordinarily, the problem to be solved is to find the neutron density at any point within the region bounded by the outermost reflector and to find the critical size of the pile, after the dimensions of the reflectors have been assigned. In all three cases mentioned the general solution of the set within each region is easily written down, even where the coefficients are variable. Thereafter, a difficulty arises in that the location of the boundaries is not assigned but is dependent upon the size of the

pile. Thus, a procedure for the solution of the problem is to solve the equations sequentially from the pile through all of the reflectors in terms of a fixed but unspecified critical size (radius or half-width) until the outer boundary is reached. Then, the outer boundary conditions together with requirements of symmetry within the pile provide sufficient specifications for the determination of critical pile size and the complete solution of the problem.

In a previous report, Mon P-202, the two-group theory for the case of multiple reflectors, without multiplication in the reflectors, and an infinite slab pile is discussed. In the determination of critical size it is found that the contribution of each reflector to the solution of the problem is independent of pile size. The equation which finally yields the critical size involves a single fourth order determinant, whose first two columns only involve the pile size, and whose last two columns are obtainable as a product of $\nu - 1$ fourth order matrices and one 4×2 matrix (four rows and two columns), all independent of pile size.

In this paper the problem is discussed from a more general point of view which includes the consideration of possible multiplication in the reflectors and three different pile shapes. First of all, the system of n second order differential equations is replaced by a system of 2n first order differential

equations. This procedure embraces all of the advantages of the first study while further providing better general perspective on the problem and simplifying somewhat certain computational aspects of the problem. The simplicity first achieved in the infinite slab case is preserved in the new procedure, but does not persist in the spherical and cylindrical cases for either method. However, in all cases it is possible to reduce the order of the fundamental determinant for the determination of critical pile size to exactly the number of groups involved in the problem.

In § 2 a special example is provided for the case of two groups and an arbitrary number of reflectors in an infinite slab pile, with multiplication in the reflectors. In § 3 the case of the infinite slab pile is treated with full generality. General treatments of the spherical and cylindrical cases are given in § 4 and § 5 respectively. Finally, in § 6, a study is made of the limiting situation in the case of the infinite slab pile and the spherical pile to provide an n - group theory with a reflector of variable density.

2. The two-group, multi-reflector problem for the infinite slab pile.

2.1. The differential equations and boundary conditions.

Let y, with an appropriate subscript to represent a particular group, represent the flux of the neutrons of this group in any of

the $\nu + 1$ regions consisting of the pile and the reflectors. Let y_1 represent the flux of thermal neutrons and y_2 the flux of non-thermal neutrons. The equations in any region have the form

$$(2.1.1)\quad \begin{aligned} \lambda_1 \frac{d^2 y_1}{dx^2} - \lambda_1 \kappa_1^2 y_1 + \lambda_2 \kappa_2^2 y_2 &= 0 , \\ \lambda_2 \frac{d^2 y_2}{dx^2} - \lambda_2 \kappa_2^2 y_2 + k \lambda_1 \kappa_1 y_1 &= 0 , \end{aligned}$$

where the constant k is zero in a reflector without multiplication.

The boundary conditions prescribe the continuity of y_i and $\lambda_i \frac{dy_i}{dx}$, for $i = 1, 2$, at the intermediate boundaries and the vanishing of y_i at the outer boundary.

2.2. <u>Reduction to a system of first order differential equations</u>. If in (2.1.1) the substitutions

$$y_3 = \lambda_1 y_1' ,$$

$$y_4 = \lambda_2 y_2'$$

are made, the system of first order equations which follow is obtained:

$$
(2.2.1)\qquad
\begin{aligned}
y_1' &= \lambda_1^{-1}\, y_3 \,, \\
y_2' &= \lambda_2^{-1}\, y_4 \,, \\
y_3' &= \lambda_1 \kappa_1^2\, y_1 - \lambda_2 \kappa_2^2\, y_2 \,, \\
y_4' &= -k\lambda_1 \kappa_1^2\, y_1 + \lambda_2 \kappa_2^2\, y_2 \,.
\end{aligned}
$$

Now, let the symbol y without a subscript denote the column vector of components y_1, y_2, y_3, y_4. Then, the equations (2.2.1) can be written in the matrix form

$$y' = My \,,$$

where

$$
M = \begin{pmatrix}
0 & 0 & \lambda_1^{-1} & 0 \\
0 & 0 & 0 & \lambda_2^{-1} \\
\lambda_1 \kappa_1^2 & -\lambda_2 \kappa_2^2 & 0 & 0 \\
-k\lambda_1 \kappa_1^2 & \lambda_2 \kappa_2^2 & 0 & 0
\end{pmatrix} .
$$

<u>Solution of the problem for the case of no multiplication in the reflectors.</u> Solutions of the form $y_i = A_i\, e^{\mu x}$, $i = 1, 2, 3, 4$, are now sought. Substitution in (2.2.1) yields the equations

$$A_1\mu = \lambda_1^{-1} A_3 ,$$

$$A_2\mu = \lambda_2^{-1} A_4 ,$$

$$A_3\mu = \lambda_1 K_1^2 A_1 - \lambda_2 K_2^2 A_2 ,$$

$$A_4\mu = -k\lambda_1 K_1^2 A_1 + \lambda_2 K_2^2 A_2 ,$$

which have a non-trivial solution in A_1, A_2, A_3, A_4 provided that

$$(2.3.1) \quad \begin{vmatrix} -\mu & 0 & \lambda_1^{-1} & 0 \\ 0 & -\mu & 0 & \lambda_2^{-1} \\ \lambda_1 K_1^2 & -\lambda_2 K_2^2 & -\mu & 0 \\ -k\lambda_1 K_1^2 & \lambda_2 K_2^2 & 0 & -\mu \end{vmatrix} = 0 .$$

This is the characteristic equation of the system (2.2.1).

The equation (2.3.1) can be written in the form

$$\begin{vmatrix} 0 & 0 & \lambda_1^{-1} & 0 \\ 0 & 0 & 0 & \lambda_2^{-1} \\ \lambda_1(K_1^2 - \mu^2) & -\lambda_2 K_2^2 & -\mu & 0 \\ -k\lambda_1 K_1^2 & \lambda_2(K_2^2 - \mu^2) & 0 & -\mu \end{vmatrix} = 0 .$$

and consequently

$$\begin{vmatrix} \kappa_1^2 - \mu^2 & -\kappa_2^2 \\ -k\kappa_1^2 & \kappa_2^2 - \mu^2 \end{vmatrix} = 0 \; .$$

or

$$(2.3.2) \qquad (\kappa_1^2 - \mu^2)(\kappa_2^2 - \mu^2) - k\,\kappa_2^2\,\kappa_2^2 = 0 \, .$$

In the pile $k > 1$ and the equation (2.3.2) in μ^2 has one positive and one negative root. Thus, the four roots of (2.3.2) may be written in the form $\pm i\,\mu_1$, $\pm\mu_2$, where μ_1 and μ_2 are real and positive numbers. In this section it is assumed that $k = 0$ in the reflectors, and the corresponding roots of (2.3.2) are $\pm\kappa_1$, $\pm\kappa_2$. In either case there exists a set of four independent solutions of the system (2.2 1). Let each solution of any independent set of solutions be made a column of a 4×4 matrix Y. To illustrate, a matrix Y_α corresponding to the α - th region* of the pile is constructed. One uses a solution of the form $y_1 = A_1 e^{\kappa_1 x}$, taking $A_1 = 1$, in the equations (2.2.1) to get the first column in Y_α. In this case it is found that $A_3 = A_4 = 0$. The substitution $y_1 = A_1 e^{-\kappa_1 x}$ gives the second column, and so on. The final result for $\alpha > 0$ is

---

*Until further notice in this section a subscript shall henceforth designate a particular region instead of a particular group.

$$
Y_\alpha(x) =
\begin{pmatrix}
e^{\kappa_1 x} & e^{-\kappa_1 x} & e^{\kappa_2 x} & e^{-\kappa_2 x} \\
0 & 0 & \lambda_1 \lambda_2^{-1} \kappa_2^{-2} (\kappa_1^2 - \kappa_2^2) e^{\kappa_2 x} & \lambda_1 \lambda_2^{-1} \kappa_2^{-2} (\kappa_1^2 - \kappa_2^2) e^{-\kappa_2 x} \\
\lambda_1 \kappa_1 e^{\kappa_1 x} & -\lambda_1 \kappa_1 e^{-\kappa_1 x} & \lambda_1 \kappa_2 e^{\kappa_2 x} & -\lambda_1 \kappa_2 e^{-\kappa_2 x} \\
0 & 0 & \lambda_1 \kappa_2^{-1} (\kappa_1^2 - \kappa_2^2) e^{\kappa_2 x} & -\lambda_1 \kappa_2^{-1} (\kappa_1^2 - \kappa_2^2) e^{-\kappa_2 x}
\end{pmatrix}
$$

which is a non-singular matrix. By taking appropriate linear combinations of the columns of this matrix one may replace the above matrix Y_α by a new one

$$
Y_\alpha(x) =
\begin{pmatrix}
\operatorname{sh} \kappa_1 x & \operatorname{sh} \kappa_2 x & \operatorname{ch} \kappa_1 x & \operatorname{ch} \kappa_2 x \\
0 & s_1 \operatorname{sh} \kappa_2 x & 0 & s_1 \operatorname{ch} \kappa_2 x \\
\lambda_1 \kappa_1 \operatorname{ch} \kappa_1 x & \lambda_1 \kappa_2 \operatorname{ch} \kappa_2 x & \lambda_1 \kappa_1 \operatorname{sh} \kappa_1 x & \lambda_1 \kappa_2 \operatorname{sh} \kappa_2 x \\
0 & \lambda_2 s_1 \kappa_2 \operatorname{ch} \kappa_2 x & 0 & \lambda_2 s_1 \kappa_2 \operatorname{sh} \kappa_2 x
\end{pmatrix}.
$$

where

$$
s_1 = \lambda_1 \lambda_2^{-1} \kappa_2^{-2} (\kappa_1^2 - \kappa_2^2).
$$

For later reference the matrix Y_o, corresponding to the pile, is computed to be

$$Y_o(x) =$$

$$\begin{pmatrix} \cos\mu_1 x & \mathrm{ch}\,\mu_2 x & \sin\mu_1 x & \mathrm{sh}\,\mu_2 x \\ r_1\cos\mu_1 x & r_2\,\mathrm{ch}\,\mu_2 x & r_1\sin\mu_1 x & r_2\,\mathrm{sh}\,\mu_2 x \\ -\lambda_1\mu_1\sin\mu_1 x & \lambda_1\mu_2\,\mathrm{sh}\,\mu_2 x & \lambda_1\mu_1\cos\mu_1 x & \lambda_1\mu_2\,\mathrm{ch}\,\mu_2 x \\ -r_1\lambda_2\mu_1\sin\mu_1 x & r_2\lambda_2\mu_2\,\mathrm{sh}\,\mu_2 x & r_1\lambda_2\mu_1\cos\mu_1 x & r_2\lambda_2\mu_2\,\mathrm{ch}\,\mu_2 x \end{pmatrix}$$

where

$$r_1 = \lambda_1\lambda_2^{-1}\kappa_2^{-2}(\kappa_1^2+\mu_1^2)\ ,$$

$$r_2 = \lambda_1\lambda_2^{-1}\kappa_2^{-2}(\kappa_1^2-\mu_2^2)\ .$$

One bears in mind that the parameters κ and λ which occur here are those characterising the material of the pile and are distinct from those occurring in the matrix Y_α written above.

In general, then, the matrix $Y(x)$ for each region is a non-singular matrix of functions of x satisfying the matrix equation

$$Y' = MY.$$

Since M is a constant matrix the related system of equations have constant coefficients. It is then clear that if in the matrix

$Y(x)$, the independent variable x is everywhere replaced by $x - x_0$, the resulting matrix $Y(x - x_0)$ is also a matrix of independent solutions if x_0 designates any point within or on a boundary of the region (pile or reflector) over which the differential equations are defined. Indeed, the replacement of x by $x - x_0$ is merely a transformation of the independent variable which has no effect upon the form of the equations. Moreover, every solution of the equations is expressible in the form $y = Yc$, where c is a column vector of four elements.

One now takes the origin as the center of the pile and $x_{\alpha - 1}$ and x_α as the bounds of the α - th region. Accordingly, the matrix Y_α may be taken as a function of $x - x_{\alpha - 1}$, that is, of the distance from the inner boundary of the region to an arbitrary point of the region.

The conditions of continuity at the pile and reflector boundaries require that

$$y_\alpha(x_\alpha - x_{\alpha - 1}) = y_{\alpha + 1}(0) \quad .$$

or, if

$$t_\alpha = x_\alpha - x_{\alpha - 1} \quad ,$$

$$y_\alpha(t_\alpha) = y_{\alpha + 1}(0) \quad .$$

Notice that in the last equation neither x_α nor $x_{\alpha - 1}$ appears, but only the thickness t_α of the α - th region.

Suppose now that a solution y_0 were given for the pile.

This could be expressed in the form

$$y_0 = Y_0 \; c_0$$

where Y_0 is the particular matrix of fundamental solutions which has been chosen within the region of the pile. Now, the solution $y_1 = Y_1 \; c_1$ in the first reflector must satisfy the relation

$$y_1(0) = y_0(a)$$

where $a = t_0$ is the half-thickness of the pile.

Hence,

$$Y_1(0)c_1 = Y_0(a)c_0 \quad ,$$

and, since all of the Y_α 's have been chosen non-singular, one obtains

$$c_1 = Y_1^{-1}(0)Y_0(a)c_0 \quad .$$

Again, at the next boundary,

$$y_2(0) = y_1(t_1) \quad ,$$

or

$$Y_2(0)c_2 = Y_1(t_1)c_1 \quad ,$$

so that

$$c_2 = Y_2^{-1}(0)Y_1(t_1)c_1 \quad .$$

or

$$c_2 = Y_2^{-1}(0)Y_1(t_1)Y_1^{-1}(0)Y_o(a)c_o$$

If this procedure is continued one obtains

$$(2.3.4)\quad c_\alpha = Y_\alpha^{-1}(0)Y_{\alpha-1}(t_{\alpha-1})Y_{\alpha-1}^{-1}(0)\cdots Y_o(a)c_o \; .$$

Thus, if the vector c_o (or the solution y_o within the pile) and the half-thickness $\underline{a}$ of the pile were known, the solution within all of the reflectors could be obtained.

The situation last considered does not generally arise, and both y_o and $\underline{a}$ must be obtained from other conditions. The physical situation requires that the solution be symmetric within the pile. Accordingly, the last two components of the vector y_o must vanish at the outer boundary; that is, the first two components of the vector y_ν must vanish at the outer boundary. To express these requirements in symbolic form let Y_o and Y_ν be partitioned into 2 x 4 matrices:

$$Y_o = \begin{pmatrix} Y_{o1} \\ Y_{o2} \end{pmatrix} , \qquad Y_\nu = \begin{pmatrix} Y_{\nu 1} \\ Y_{\nu 2} \end{pmatrix} .$$

and then require that

$$(2.3.5)\qquad Y_{o2}(0)c_o = 0 \; .$$

$$(2.3.6)\qquad Y_{\nu 1}(t_\nu)c_\nu = 0 \; .$$

If the matrix Y_o is written in the form (2.3.3) the requirement (2.3.5)

is met by making the last two components of the vector c_0 equal to zero. The end result is the same if one drops out the last two columns of Y_0 thus making it a 4 x 2 matrix composed of the first two columns of (2.3.3), and it will be assumed that this has been done in what follows.

Now, in (2.3.4), set $\alpha = \nu$ and multiply on the left by $Y_{\nu 1}(t_\nu)$. Then it follows from (2.3.6) that

$$Y_{\nu 1}(t_\nu)Y_\nu^{-1}(0)Y_{\nu-1}(t_{\nu-1})Y_{\nu-1}^{-1}(0)\ldots Y_0(a)c_0 = 0 \quad .$$

The matrix $Y^{-1}(0)Y_{\nu-1}(t_{\nu-1})Y_{\nu-1}^{-1}(0)\ldots Y_0(a)$ is a 4 x 2 matrix since $Y_0(a)$ is a 4 x 2 matrix, and $Y_{\nu 1}(t_\nu)$ is a 2 x 4 matrix. Then the product is a 2 x 2 matrix. The equations are consistent if and only if the determinant of the coefficients vanishes:

$$\left| Y_{\nu 1}(t_\nu)Y_\nu^{-1}(0)Y_{\nu-1}(t_{\nu-1})Y_{\nu-1}^{-1}(0)\ldots Y_0(a)\right| = 0 \quad , \tag{2.3.7}$$

The determinant is of order two and the solution of the equation gives the critical pile size $\underline{a}$.

The computation is simplified in a numerical problem if $Y_\nu(x - x_{\nu-1})$ is replaced by $Y_\nu(x - x_\nu)$. This has the effect of making the last two components of the vector c_ν equal to zero, and condition (2.3.6) becomes

$$Y_{\nu 1}(0)c = 0 \quad .$$

Equation (2.3.7) is now written

$$\left| Y_{\nu 2}^{-1}(-t_\nu)\, Y_{\nu-1}(t_{\nu-1})\, Y_{\nu-1}^{-1}(0) \ldots Y_0(s) \right| = 0 \; .$$

As a matter of record the following formulas are listed:

$$Y_\nu(0) = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & s_1 \\ \lambda_1 \kappa_1 & \lambda_1 \kappa_2 & 0 & 0 \\ 0 & \lambda_2 s_1 \kappa_2 & 0 & 0 \end{pmatrix} ,$$

$$Y_\nu^{-1}(0) = \begin{pmatrix} 0 & 0 & \lambda_1^{-1}\kappa_1^{-1} & -s_1^{-1}\lambda_2^{-1}\kappa_1^{-1} \\ 0 & 0 & 0 & s_1^{-1}\lambda_2^{-1}\kappa_2^{-1} \\ 1 & -s_1^{-1} & 0 & 0 \\ 0 & s_1^{-1} & 0 & 0 \end{pmatrix} ,$$

$$Y_\nu(t_\nu) Y_\nu^{-1}(0) =$$

$$\begin{pmatrix} \operatorname{ch}\kappa_1 t_\nu & \frac{1}{s_1}(\operatorname{ch}\kappa_2 t_\nu - \operatorname{ch}\kappa_1 t_\nu) & \operatorname{sh}\kappa_1 t_\nu & \frac{1}{s_1\lambda_2}\left(\frac{\operatorname{sh}\kappa_2 t_\nu}{\kappa_2} - \frac{\operatorname{sh}\kappa_1 t_\nu}{\kappa_1}\right) \\ 0 & \operatorname{ch}\kappa_2 t_\nu & 0 & \frac{1}{\lambda_2\kappa_2}\operatorname{sh}\kappa_2 t_\nu \\ \lambda_1\kappa_1 \operatorname{sh}\kappa_1 t_\nu & \frac{\lambda_1}{s_1}(\kappa_2 \operatorname{sh}\kappa_2 t_\nu - \kappa_1 \operatorname{sh}\kappa_1 t_\nu) & \operatorname{ch}\kappa_1 t_\nu & \frac{\lambda_1}{s_1\lambda_2}(\operatorname{ch}\kappa_2 t_\nu - \operatorname{ch}\kappa_1 t_\nu) \\ 0 & \lambda_2\kappa_2 \operatorname{sh}\kappa_2 t_\nu & 0 & \operatorname{ch}\kappa_2 t_\nu \end{pmatrix} .$$

The first two rows only occur in the determinant of (2.3.7).

It should be emphasized that in the preceding formulas the subscripts on the Y's and t's refer to a particular region (pile or reflector) and the subscripts elsewhere designate a particular group. When these formulas are used for actual computation a double subscript notation (ij) should be used, where i designates the region and j designates the group.

2.4. Further observations on the solution. It is possible that the computation in the problem of the preceding section can be simplified by making the matrix $Y_\alpha(0)$ of initial values the identity matrix. It is clear that if $Y_\alpha(x - x_{\alpha-1})$ is any non-singular matrix of solutions then so also is the matrix

$$W_\alpha(x - x_{\alpha-1}) = Y_\alpha(x - x_{\alpha-1})Y_\alpha^{-1}(0) \quad ,$$

and moreover

$$W_\alpha(0) = I \quad .$$

Then, the fundamental equation (2.3.7) becomes

$$\left| W_{\nu 1}(t_\nu)W_{\nu-1}(t_{\nu-1}) \ldots W_{o2}(a) \right| = 0 \quad ,$$

where $W_{\nu 1}(t_\nu)$ consists of the first two rows of $W_\nu(t_\nu)$ and $W_{o2}(a)$ consists of the first two columns of $W_o(a)$.

In the preceding section the formula

$$(2.4.1) \quad c_N = Y_N^{-1}(0)Y_{N-1}(t_{N-1})Y_{N-1}^{-1}(0) \ldots Y_o(a)c_o = P\,c_o$$

was obtained without imposing any restriction on the matrix of

solutions Y_α beyond its non-singularity. It is now observed that (2.4.1) can be written in the equivalent form

$$c_0 = Y_0^{-1}(a)Y_1(0)Y_1^{-1}(t_1)\ .\ .\ .\ Y_N(0)c_N \equiv P^{-1}c_N$$

Accompanying conditions required that

$$(0_2\ ,\ I_2)Y_0(0)c_0 = 0\ ,$$

$$(I_2\ ,\ 0_2)Y_N(t_N)c_N = 0\ ,$$

where 0_2 and I_2 are 2 x 2 zero - and identity - matrices, respectively. By the special choice of Y_0 whereby a partitioning of Y_0 (0) has the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} .$$

the condition on the vector c_0 alone is equivalent to the requirement that the last two components of c_0 be zero. The subsequent elimination of the two vectors c_0 and c_N was found to yield a determinant of order two in the final equation of the problem. A similar special choice of Y_N is useful when the outer reflector is infinite in the x - direction. To this end one chooses the first two columns of Y_N to be a set of solutions of the differential equations which vanish at infinity. Hence, the last two components of c_N are both zero and there remains the

determinantal equation

$$(2.4.2) \qquad \left| Y_{N2}^{-1}(0) Y_{N-1}(t_{N-1}) \ldots Y_{o1}(a) \right| = 0 ,$$

where $Y_{N2}^{-1}(0)$ is the lower half of the matrix $Y_N^{-1}(0)$ and $Y_{o1}(a)$ the left-hand half of the matrix $Y_o(a)$.

A procedure which simplifies the computation of $Y_{N2}^{-1}(0)$ in equation (2.4.2) has been suggested by B. Spinrad. A special choice of the last two columns in Y_N is made so that the task of computing $Y_N^{-1}(0)$ is made as simple as possible. One writes $Y_N(x)$ in the form

$$Y_N(x) = \begin{pmatrix} e^{-\kappa_1 x} & e^{-\kappa_2 x} & \frac{sh\,\kappa_1 x}{\lambda_1 \kappa_1} & \frac{1}{\lambda_2 s_1 \kappa_1 \kappa_2}(\kappa_1 sh\,\kappa_2 x - \kappa_2 sh\,\kappa_1 x) \\ 0 & s_1 e^{-\kappa_2 x} & 0 & \frac{sh\,\kappa_2 x}{\lambda_2 \kappa_2} \\ -\lambda_1 \kappa_1 e^{-\kappa_1 x} & -\lambda_1 \kappa_2 e^{-\kappa_2 x} & ch\,\kappa_1 x & \frac{\lambda_1}{\lambda_2 s_1}(ch\,\kappa_2 x - ch\,\kappa_1 x) \\ 0 & -\lambda_2 s_1 \kappa_2 e^{-\kappa_2 x} & 0 & ch\,\kappa_2 x \end{pmatrix},$$

so that $Y_N(0)$ takes the form

$$\begin{pmatrix} A & 0 \\ B & I \end{pmatrix},$$

the inverse of which is readily computed. It is the lower half of this inverse which is used in (2.4.2).

In case the pile is not symmetric there is no requirement of symmetry at the center. Instead, if O and N designate the two outer reflectors, it is required that

$$(I_2, O_2)Y_o(0)c_o = 0 \quad ,$$

$$(I_2, O_2)Y_N(t_N)c_N = 0 \quad ,$$

provided that both outer reflectors are finite. The special choice of both Y_o and Y_N results in $Y_o(0)$ and $Y_N(t_N)$ having the partitioned form

$$\begin{pmatrix} 0 & B \\ A & C \end{pmatrix} .$$

This means that the first two columns of each vanish in their first two components at the outer boundaries. In case either reflector is infinite in the x - direction the requirements on $Y_o(0)$ or $Y_N(t_N)$ are meaningless, but in this case the first two columns of the Y matrix are chosen to vanish at infinity. In either case, the last two components of c_o and c_N are zero. Note that if the zero - reflector is infinite, the origin must be moved to its inner boundary. Now, if c_{o1} and c_{N1} denote the two-vectors of non-zero components there results either of the equivalent sets of 2 x 2 equations of the form

$$Qc_{o1} = 0 \quad , \quad Rc_{N1} = 0 \quad .$$

from which one obtains either of the equivalent determinantal equations

$$Q = 0, \quad R = 0 .$$

If it is assumed for the sake of simplicity that all of the reflectors are finite, note that it makes little difference which of the N regions is left with its thickness undetermined. The unknown thickness t_α is involved in only a single matrix of the product Q or R. This matrix is multiplied on the left by a constant matrix of dimensions 2 x 4 and on the right by one of dimensions 4 x 2.

2.5. Multiplication in the reflectors. In this section the case of multiplication in the reflectors is considered. If $k = 0$ in the characteristic equation (2.3.2) there results the foregoing situation in which there are two positive values of μ^2. If $0 < k < 1$ the μ^2's are still positive and the matrix Y_ν is as before. If $k = 1$, there is one positive μ^2 and one zero μ^2. The matrix Y_α takes the form

$$Y_\alpha = \begin{pmatrix} 1 & x & e^{(\kappa_1^2+\kappa_2^2)x} & e^{-(\kappa_1^2+\kappa_2^2)x} \\ \dfrac{\lambda_1\kappa_1^2}{\lambda_2\kappa_2^2} & \dfrac{\lambda_1\kappa_1^2 x}{\lambda_2\kappa_2^2} & re^{(\kappa_1^2+\kappa_2^2)x} & re^{-(\kappa_1^2+\kappa_2^2)x} \\ 0 & \lambda_1 x & se^{(\kappa_1^2+\kappa_2^2)x} & -se^{-(\kappa_1^2+\kappa_2^2)x} \\ 0 & \dfrac{\lambda_1\kappa_1^2 x}{\kappa_2^2} & -se^{(\kappa_1^2+\kappa_2^2)x} & se^{-(\kappa_1^2+\kappa_2^2)x} \end{pmatrix} .$$

or

$$
Y_\alpha =
\begin{pmatrix}
x & \operatorname{sh}(\kappa_1^2+\kappa_2^2)x & 1 & \operatorname{ch}(\kappa_1^2+\kappa_2^2)x \\
\dfrac{\lambda_1\kappa_1^2 x}{\lambda_2\kappa_2^2} & r\,\operatorname{sh}(\kappa_1^2+\kappa_2^2)x & \dfrac{\lambda_1\kappa_1^2}{\lambda_2\kappa_2^2} & r\,\operatorname{ch}(\kappa_1^2+\kappa_2^2)x \\
\lambda_1 & s\,\operatorname{ch}(\kappa_1^2+\kappa_2^2)x & 0 & s\,\operatorname{sh}(\kappa_1^2+\kappa_2^2)x \\
\dfrac{\lambda_1\kappa_1^2}{\kappa_2^2} & -s\,\operatorname{ch}(\kappa_1^2+\kappa_2^2)x & 0 & -s\,\operatorname{sh}(\kappa_1^2+\kappa_2^2)x
\end{pmatrix} .
$$

where

$$
r = \frac{\lambda_1}{\lambda_2\kappa_2^2}\left[\kappa_1^2 - (\kappa_1^2+\kappa_2^2)\right] ,
$$

$$
s = \lambda_1(\kappa_1^2+\kappa_2^2) .
$$

If $k > 1$, $Y_\alpha(x)$ takes the form of $Y_0(x)$ as in (2.3.3).

3. <u>The multi-group, multi-reflector problem for the infinite slab pile.</u>

3.1. <u>The general formulation.</u> As in § 2, let y, with an appropriate subscript to represent a particular group, represent the flux of the neutrons of this group in any of the $\nu + 1$ regions consisting of the pile and the reflector. The subscripts are chosen so as to increase with the mean energy of the group, that is, y_1 represents the flux of thermal neutrons, y_2 that of the slowest group of fast neutrons, and y_n that of the fastest group (the fission neutrons). The differential equations in any region have the form

$$
\begin{aligned}
&\lambda_i \nabla^2 y_i - \lambda_i \kappa_i^2 y_i + \lambda_{i+1} \kappa_{i+1}^2 y_{i+1} = 0 \;, \qquad (i < n) \;, \\
(3.1.1) \quad & \\
&\lambda_n \nabla^2 y_n + k \lambda_1 \kappa_1^2 y_1 - \lambda_n \kappa_n^2 y_n = 0 \;,
\end{aligned}
$$

where the constant k is zero in a reflector without multiplication.

The boundary conditions prescribe the continuity of y_i and of $\lambda_i \nabla y_i$ at the intermediate boundaries and the vanishing of y_i at the outer boundary or boundaries.

3.2. The infinite slab pile. For the infinite slab pile the differential equations are especially simple in form since $\nabla^2 y_i$ becomes merely the second derivative with respect to x and all coefficients of the y_i and their derivatives are constant. It is convenient to introduce the additional variables

$$
y_{n+j} = \lambda_j y_j' \;, \qquad (j \leq n) \;,
$$

which makes it possible to replace the system of n second order equations by the 2n first order equations

$$
\begin{aligned}
& y_j' = \lambda_j^{-1} y_{n+j} \;, && (j \leq n) \;, \\
(3.2.1) \quad & y_{n+i}' = \lambda_i \kappa_i^2 y_i - \lambda_{i+1} \kappa_{i+1}^2 y_{i+1} \;, && (i < n) \;, \\
& y_{2n}' = -k \lambda_1 \kappa_1^2 y_1 + \lambda_n \kappa_n^2 y_n \;. &&
\end{aligned}
$$

If the symbol y without a subscript denotes the column vector of components $y_1, y_2, \ldots, y_{2n}$, these equations can be written in the matrix form

(3.2.2) $$y' = My \quad ,$$

where the 2n x 2n matrix M can be partitioned in the form

$$M = \begin{pmatrix} 0 & \Lambda^{-1} \\ M_1 & 0 \end{pmatrix} ,$$

with

$$\Lambda^{-1} = \begin{pmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \lambda_n^{-1} \end{pmatrix} ,$$

and

$$M_1 = \begin{pmatrix} \lambda_1\kappa_1^2 & -\lambda_2\kappa_2^2 & 0 & \cdots & 0 \\ 0 & \lambda_2\kappa_2^2 & -\lambda_3\kappa_3^2 & \cdots & 0 \\ \cdot & & \cdot & \cdot & \cdot \\ -\lambda_1\kappa_1^2 & 0 & 0 & \cdots & \lambda_n\kappa_n^2 \end{pmatrix} .$$

The characteristic equation

$$|M - \mu I| = 0$$

of the equation (3.2.2) is easily written down with the aid of the relation

$$\begin{pmatrix} -\mu I & \Lambda^{-1} \\ M_1 & -\mu I \end{pmatrix} \begin{pmatrix} I & 0 \\ \mu\Lambda & I \end{pmatrix} = \begin{pmatrix} 0 & \Lambda^{-1} \\ M_1 - \mu^2\Lambda & -\mu I \end{pmatrix} ,$$

where the first matrix on the left is $M - \mu I$ and the second has the determinant unity. Hence

$$|M - \mu I| = |M_1 - \mu^2\Lambda|\,|\Lambda^{-1}| = |M_1\Lambda^{-1} - \mu^2 I|$$

$$= \begin{vmatrix} \kappa_1^2 - \mu^2 & -\kappa_2^2 & 0 & \cdots & 0 \\ 0 & \kappa_2^2 - \mu^2 & -\kappa_3^2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -k\,\kappa_1^2 & 0 & 0 & \cdots & \kappa_n^2 - \mu^2 \end{vmatrix} .$$

The last determinant can be expanded immediately to yield the equation

$$(3.2.3) \quad (\kappa_1^2 - \mu^2)(\kappa_2^2 - \mu^2) \cdots (\kappa_n^2 - \mu^2) = k\,\kappa_1^2\kappa_2^2 \cdots \kappa_n^2 .$$

It is clear that when $k = 0$ the roots of the characteristic equation are $\pm\kappa_i$. To investigate the case $k > 0$ consider the graph of the left member of (3.2.3) plotted against μ^2. This crosses the μ^2 - axis at κ_i^2 and crosses the vertical axis at $\kappa_1^2\kappa_2^2 \cdots \kappa_n^2$. The effect of subtracting $k\,\kappa_1^2\kappa_2^2 \cdots \kappa_n^2$

from the function is to raise the μ^2 - axis by this amount. Hence, if $k < 1$ the real roots of the equation in μ^2 are all positive; if $k = 0$ there is one zero root and all other real roots are positive; if $k > 1$ there is one negative root and all other real roots are positive. However, one cannot be assured in general, for $n > 2$, that the roots in μ^2 are all real since, for any $k > 0$, by making the difference between any pair of κ's sufficiently small one may introduce a pair of complex roots.

Whatever the nature of the characteristic roots, one can always employ them to write down a set of $2n$ independent solutions of the differential equations. If each of these solutions is made a column of a $2n \times 2n$ matrix Y, as in § 2, then Y is a non-singular matrix satisfying the matrix equation

$$Y'' = MY \quad .$$

From this point on the discussion for the case of two groups given in § 2 applies without significant changes for the case of n groups. The equation for the determination of critical pile size for all of the situations considered will involve a determinant of order exactly n . The fundamental determinantal equation may of course be used to find k if the pile and reflector dimensions are fixed.

4. <u>The multi-group, multi-reflector problem for the spherical pile.</u>

4.1. The differential equations and boundary conditions. If the spherical pile has reflectors in the form of spherical shells, such as to provide complete spherical symmetry, then

$$r \nabla^2 y = \frac{d^2(ry)}{dr^2} \quad ,$$

Then the substitution

$$\left.\begin{aligned} v_i &= r y_i \quad , \\ v_{n+i} &= \lambda_i v_i' \quad , \end{aligned}\right\} \qquad (i \leq n) \quad ,$$

in the differential equations (3.1.1) yields 2n differential equations in v_i, v_{n+i} of exactly the same form as those in y_i, y_{n+i} for the slab pile. However, in the spherical case, the boundary conditions which impose continuity upon y_i and $\lambda_i y_i'$, when expressed in terms of v_i, require the continuity of v_i and

$$\lambda_i \frac{d(v_i/r)}{dr} = \frac{v_{n+i}}{r} - \frac{\lambda_i v_i}{r^2}$$

at the pile and reflector boundaries. Note in passing that when the λ_i's are equal the v_{n+i}'s are also continuous and the procedure is in all respects similar to the slab case.

4.2. Solution for the general case. The formation of a fundamental set of solutions v constituting a non-singular

matrix V does not differ in any respect from the formation of such a matrix Y in the case of the slab pile. These solutions can be expressed as functions of r or of $\rho = r - r_{\alpha - 1}$, where α designates the α - th region, since any fundamental set expressed as functions of r remains a fundamental set when r is replaced throughout by ρ. In this procedure it must be noted that one may obtain solutions $y_i = v_i/r$ which become infinite under certain circumstances.

It is supposed now that in each region α a matrix of solutions V_α has been found and, for the sake of convenience, that this is so chosen that

$$V_\alpha (r_{\alpha - 1}) = I, \quad (\alpha = 1, 2, \ldots, \nu) \quad .$$

Since none of the r_α's is known until the critical radius $r_0 = a$ is found, this choice involves expressing the solutions v as functions of $\rho = r - r_{\alpha - 1}$ and choosing V_α to be the identity matrix when $\rho = 0$.

In the pile ($\alpha = 0$) y must remain finite at $r = 0$, so that out of the 2n solutions n can be eliminated Accordingly, the required solution is of the form

$$v_0 = V_{01} \, c_0 \quad ,$$

where c_0 is an n - vector of constants and V_{01} is the 2n x n matrix of v's (expressed in terms of trigonometric and hyperbolic

sines) whose first n rows vanish at $r = 0$. The rank of V_{01} must be n.

The continuity requirement involves the continuous vector

$$(4.2.1) \quad \begin{pmatrix} I & 0 \\ -r^{-2}\Lambda & r^{-1} I \end{pmatrix} v =$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ -\frac{\lambda_1}{r^2} & 0 & \cdots & 0 & \frac{1}{r} & 0 & \cdots & 0 \\ 0 & -\frac{\lambda_2}{r^2} & \cdots & 0 & 0 & \frac{1}{r} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & -\frac{\lambda_n}{r^2} & 0 & 0 & \cdots & \frac{1}{r} \end{pmatrix} v .$$

Thus, it is first of all required that

$$\begin{pmatrix} I & 0 \\ -a^{-2}\Lambda_0 & a^{-1} I \end{pmatrix} V_{01}(a) c_0 = \begin{pmatrix} I & 0 \\ -a^{-2}\Lambda_1 & a^{-1} I \end{pmatrix} c_1 .$$

provided that $V_1 = I$ at $r = r_0 = a$.

Note that

$$\begin{pmatrix} I & 0 \\ a^{-1}\Lambda_1 & a\,I \end{pmatrix}\begin{pmatrix} I & 0 \\ -a^{-2}\Lambda_1 & a^{-1}\,I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

so that (4.2.1) may be written as

$$\begin{pmatrix} I & 0 \\ a^{-1}\Lambda_1 & a\,I \end{pmatrix}\begin{pmatrix} I & 0 \\ -a^{-2}\Lambda_0 & a^{-1}\,I \end{pmatrix} V_{01}(a)c_0 = c_1 \quad ,$$

or

$$(4.2.2)\quad \begin{pmatrix} I & 0 \\ a^{-1}\,(\Lambda_1 - \Lambda_0) & I \end{pmatrix} V_{01}(a)c_0 = c_1 \quad .$$

Next, if the notation $t_\alpha = r_\alpha - r_{\alpha-1}$ is introduced and if all v_α's are expressed as functions of $\rho = r - r_{\alpha-1}$, the continuity requirement at the second boundary yields

$$\begin{pmatrix} I & 0 \\ -(a+t_1)^{-2}\Lambda_1 & (a+t_1)^{-1}I \end{pmatrix} V_1(t_1)c_1 = \begin{pmatrix} I & 0 \\ -(a+t_1)^{-2}\Lambda_2 & (a+t_1)^{-1}I \end{pmatrix} c_2 \quad ,$$

or

$$(4.2.3)\quad \begin{pmatrix} I & 0 \\ (a+t_1)^{-1}(\Lambda_2-\Lambda_1) & I \end{pmatrix} V_1(t_1)c_1 = c_2 \quad .$$

Then from (4.2.2) and (4.2.3) it follows that

$$\begin{pmatrix} I & 0 \\ (a+t_1)^{-1}(\Lambda_2-\Lambda_1) & I \end{pmatrix} V_1(t_1) \begin{pmatrix} I & 0 \\ a^{-1}(\Lambda_1-\Lambda_0) & I \end{pmatrix} V_{01}(a)c_0 = c_2 \quad .$$

The next step gives

$$\begin{pmatrix} I & 0 \\ (a+t_1+t_2)^{-1}(\Lambda_3-\Lambda_2) & I \end{pmatrix} V_2(t_2)c_2 = c_3 \quad ,$$

which is used to eliminate c_2 in the last two equations. If this procedure is continued one obtains finally

$$(4.2.4)\quad \begin{pmatrix} I & 0 \\ \xi_{\nu-1}^{-1}(\Lambda_\nu-\Lambda_{\nu-1}) & I \end{pmatrix} V_{\nu-1}(t_\nu) \cdots \begin{pmatrix} I & 0 \\ a^{-1}(\Lambda_1-\Lambda_0) & I \end{pmatrix} V_{01}(a)c_0 = c_\nu$$

where $\qquad \xi_\nu = a + t_1 + t_2 + \cdots + t_\nu \quad .$

This matrix equation expresses the 2n components of c_ν in terms of the n components of c_0 and the unknown critical radius $\underline{a}$. An additional matrix relation is needed for the elimination of c_0, and this is found in the requirement that the solution v_ν shall vanish in its first n components at the outer boundary.

Now, $v_\nu = V_\nu c_\nu$, whence, if $V_{\nu 1}$ denotes the matrix of the first n rows of V_ν, the condition in question takes the form

$$V_{\nu 1}(t_\nu)c_\nu = 0 .$$

Accordingly, if the equation (4.2.4) is multiplied on the left by $V_{\nu 1}$, there results the equation

$$V_{\nu 1}(t_\nu)\begin{pmatrix} I & 0 \\ \xi_{\nu-1}^{-1}(\Lambda_\nu - \Lambda_{\nu-1}) & I \end{pmatrix} V_{\nu-1}(t_{\nu-1}) \cdots \begin{pmatrix} I & 0 \\ \xi_0^{-1}(\Lambda_1 - \Lambda_0) & I \end{pmatrix} V_{01}(a)c_0 = 0 .$$

The matrix coefficient of c_0 is n x n and its determinant must vanish, that is,

$$\left| V_{\nu 1}(t_\nu) \cdots V_{01}(a) \right| = 0 .$$

Note again that it has been presupposed that $V_\alpha(0) = I$. If this normalization is not made the matrices $V_\alpha^{-1}(0)$ must also be included in the matrix product. Note also that a does not occur in any of the V_α's for $\alpha > 0$ but only in the elements of the form $(a + t_1 + \cdots + t_\alpha)^{-1}$ multiplied by a difference of Λ's in consecutive regions. Note finally that $V_\alpha(t_\alpha)$ is of the same form as in the slab case.

5. The multi-group, multi-reflector problem for the infinite cylindrical pile.

5.1. The differential equations and boundary conditions. If the cylindrical pile has reflectors in the form of cylindrical shells, such as to provide axial symmetry, then

$$\nabla^2 y = \frac{d^2 y}{dr^2} + \frac{1}{r}\frac{dy}{dr} .$$

The differential equations in any region may be written in the form

$$\lambda_i \frac{d}{dr}\left(r \frac{dy_i}{dr}\right) - \lambda_i \kappa_i^2 r y_i + \lambda_{i+1} \kappa_{i+1}^2 r y_{i+1} = 0 , \qquad (i < n),$$

$$\lambda_n \frac{d}{dr}\left(r \frac{dy_n}{dr}\right) + k \lambda_1 \kappa_1^2 r y_1 - \lambda_n \kappa_n^2 r y_n = 0 .$$

The boundary conditions require the continuity of y_i and $\lambda_i \frac{dy_i}{dr}$ at the intermediate boundaries and the vanishing of y_i at the outer boundary or boundaries.

5.2. Solution for the general case. The introduction of the variables

$$y_{n+j} = \lambda_j r y_j' , \qquad (j \leq n),$$

has the effect of replacing the system of n second order equations (5.1.1) by the 2n first order equations

$$y_i' = \lambda_i^{-1} r^{-1} y_{n+i} , \qquad (i \leq n),$$

$$(5.2.2) \quad y_{n+i}' = \lambda_i \kappa_i^2 r y_i - \lambda_{i+1} \kappa_{i+1}^2 r y_{i+1} , \qquad (i < n),$$

$$y_{2n}' = -k \lambda_1 \kappa_1^2 r y_1 + \lambda_n \kappa_n^2 r y_n .$$

If the symbol y without a subscript denotes the column vector of components $y_1, y_2, \ldots, y_n$, then the equations (5.2.2) can be written in the matrix form $y'' = My$, where M is the non-constant matrix of coefficients in the equations.

Solutions of the system (5.2.2) of the form

$$(5.2.3) \qquad \begin{aligned} y_i &= a_i\, J_0(\mu r) , \\ y_{n+i} &= a_{n+i}\, r\, J_1(\mu r) , \end{aligned} \qquad (i \leq n),$$

are now sought. Substitution of (5.2.3) into (5.2.2) gives

$$(5.2.4) \qquad \begin{aligned} -a_i\mu &= \lambda_i^{-1} a_{n+i} , && (i \leq n), \\ a_{n+i}\mu &= \lambda_i \kappa_i^2 a_i - \lambda_{i+1}\kappa_{i+1}^2 a_{i+1} , && (i < n), \\ a_{2n}\mu &= -k\lambda_1\kappa_1^2 a_1 + \lambda_n\kappa_n^2 a_n . \end{aligned}$$

These equations have a non-trivial solution in a_i provided that

$$(5.2.5) \qquad \begin{vmatrix} -\mu I & \Lambda^{-1} \\ T_1 & -\mu I \end{vmatrix} = 0 ,$$

where

$$\Lambda^{-1} = \begin{pmatrix} -\lambda_1^{-1} & 0 & \ldots & 0 \\ 0 & -\lambda_2^{-1} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & -\lambda_n^{-1} \end{pmatrix} ,$$

and

$$K_1 = \begin{pmatrix} \lambda_1\kappa_1^2 & -\lambda_2\kappa_2^2 & 0 & \cdots & 0 \\ 0 & \lambda_2\kappa_2^2 & -\lambda_3\kappa_3^2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -k\lambda_1\kappa_1^2 & 0 & 0 & \cdots & \lambda_n\kappa_n^2 \end{pmatrix}$$

By a reduction similar to that used in § 3.2 the characteristic equation (5.2.5) becomes

$$(5.2.6)\quad (\kappa_1^2+\mu^2)(\kappa_2^2+\mu^2)\ldots(\kappa_n^2+\mu^2) = k\kappa_1^2\kappa_2^2\ldots\kappa_n^2 .$$

Note that the roots of this equation in μ^2 are the negatives of the roots of (3.2.2) in μ^2.

If J_0 and J_1 are replaced by Y_0 and Y_1* respectively in (5.2.3), the system (5.2.4), and consequently the equation (5.2.6), remains unchanged. However, if the combinations I_0, I_1 or K_0, K_1 are used the effect is the expected one of changing the signs of the roots of (5.2.6) in μ^2, or of replacing μ by $i\mu$.

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* The notation for Bessel functions used in this paper is consistent with that found in Theory of Bessel Functions by G N Watson

If the roots of (5.2.6) in μ^2 are all real, then the functions J_0 and Y_0 correspond to the positive roots and the functions I_0 and K_0 to the negative roots. Thus, in this case at least, a set of 2n independent solutions of the differential equations can be found. As in the preceding sections, if each of these solutions is made a column of a 2n x 2n matrix V, then V is a non-singular matrix satisfying the matrix equation

$$V' = MV \quad .$$

In the slab and spherical cases certain material simplifications were found by expressing the solutions in the α - th region as functions of a coordinate originating at the innermost boundary of the region. This simplification is not possible in the cylindrical case since the matrix M is no longer a constant matrix. Accordingly, the determination of critical pile size even in the simplest cases becomes an extremely arduous and uninviting task. The equation which yields the critical pile size may be written as the n x n determinantal equation

$$(5.2.7) \quad \left| V_{\nu 1}(\xi_\nu) V_\nu^{-1}(\xi_{\nu-1}) V_{\nu-1}(\xi_{\nu-1}) V_{\nu-1}^{-1}(\xi_{\nu-2}) \cdots V_{01}(a) \right| = 0 \quad ,$$

where $\xi_\alpha = a + t_1 + t_2 + \cdots + t_\alpha$, V_{01} is a 2n x n matrix of v's expressed in terms of the functions J_0 and I_0, which are finite

for $r \geq 0$, and where $V_{\nu 1}$ is an $n \times 2n$ matrix, the upper half of the matrix V_ν . Note that all of the matrices in (5.2.7) involve the pile size $\underline{a}$.

6. Multi-group pile theory with continuously varying parameters in the reflector.

6.1. The case of the infinite slab pile. In this discussion it is now assumed that the λ's and κ's of the multi-group pile theory equations are continuously varying parameters. In § 3 the system of n second order differential equations was replaced with a system of 2n first order equations with the aid of the substitution

$$y_{n+j} = \lambda_j y'_j \qquad . \qquad (j \leq n),$$

where the λ_j's are now functions of . It follows then from (3.2.1) that the system

$$(\lambda_i y_i)' - \lambda_i \kappa_i^2 y_i + \lambda_{i+1} \kappa_{i+1}^2 y_{i+1} = 0 \ , \ (i < n),$$

$$(\lambda_n y_n)' - \lambda_n \kappa_n^2 y_n + k \lambda_1 \kappa_1^2 y_1 = 0 \quad ,$$

replaces the original system of n second order differential equations.

A convenient point of departure in this section is the consideration of the sequence of sets of equations

$$y'_\alpha = M_\alpha y_\alpha \quad ,$$

where y_α is the column vector of 2n components representing a solution of the equations in the α - th reflector, and M_α is a constant matrix composed of the parameters characteristic of that region. Heretofore no relation between the M's for distinct regions was assumed. The solutions y_α, however, were required in all cases to satisfy the continuity relations

$$y_\alpha (x_\alpha) = y_{\alpha+1}(x_\alpha) \quad .$$

If y is expressed in terms of the fundamental matrices Y then the continuity relations take the form

$$(6.1.1) \qquad Y_\alpha (x_\alpha) c_\alpha = Y_{\alpha+1}(x_\alpha) c_{\alpha+1} \quad .$$

It is supposed now that the pile and reflectors constitute a single region with continuously varying parameters, or parameters having at most a finite number of discontinuities. Then the several sets of equations with constant coefficients are replaced by a single set of equations

$$Y' = M(x)y$$

with varying coefficients. Since the case in which M has components with discontinuities presents no essential complications, it will suffice to consider only the continuous

case. The interval from pile boundary to reflector boundary is broken up into slabs of thickness Δx and an approximation to the single set of equations with varying coefficients is achieved with a sequence of sets with constant coefficients. This sequence of sets has the same form as the one previously considered with

$$M_\alpha = M(\xi_\alpha), \qquad x_{\alpha-1} \leq \xi_\alpha \leq x_\alpha ,$$

that is to say with M of the form taken by $M(x)$ when the components are evaluated at an arbitrary but fixed point within the α-th region.

The fundamental matrices Y_α are chosen as in § 2.4 so that

$$Y_\alpha(x_{\alpha-1}) = I \quad .$$

Then, from (5.1.1) it follows that

$$c_{\alpha+1} = Y_\alpha(x_\alpha)c_\alpha \quad .$$

The matrix $Y_\alpha(x_\alpha)$ may be expanded in powers of Δx to give

$$Y_\alpha(x_\alpha) = Y_\alpha(x_{\alpha-1} + \Delta x)$$

$$= Y_\alpha(x_{\alpha-1}) + Y_\alpha \quad {}_\alpha \quad \Delta x \quad . \; . \; .$$

$$\alpha \quad {}_{\alpha-1} \; \cdots \; {}_\alpha Y_\alpha(x_{\alpha-1}) = M_\alpha I = M_\alpha \quad .$$

then

$$Y_\alpha(x_\alpha) = I + M_\alpha \Delta x + \ldots \quad .$$

It will suffice in this discussion to keep only the first two terms of this series.

By definition $y_\alpha(x) = Y_\alpha(x)c_\alpha$. Then it follows that $y_\alpha(x_{\alpha-1}) = Y_\alpha(x_{\alpha-1})c_\alpha = Ic_\alpha = c_\alpha$. Hence, if $y(x)$ represents the solution for the entire reflector, then $y(x)$ is identified with $y_1(x_0) = c_1$, where x_0 designates the inner boundary of the first reflector of the thickness Δx. Moreover, the entire approximation to the exact solution $y(x)$ evaluated at x_α is given by

$$\begin{aligned} y(x_\alpha) \sim c_{\alpha+1} &= (I + M_\alpha \Delta x)c_\alpha \\ &= (I + M_\alpha \Delta x)(I + M_{\alpha-1}\Delta x)c_{\alpha-1} \\ &= \ldots \\ &= (I + M_\alpha \Delta x)(I + M_{\alpha-1}\Delta x)\ldots(I + M_1 \Delta x)c \quad . \end{aligned}$$

where $c = c_1$, or

$$\begin{aligned} y(x_\alpha) \sim \Big[I + \sum_{\beta}^{1,\alpha} M_\beta \Delta x + \sum_{\beta > \beta'}^{1,\alpha} M_\beta M_{\beta'} \Delta x^2 \\ + \sum_{\beta > \beta' > \beta''}^{1,\alpha} M_\beta M_{\beta'} M_{\beta''} \Delta^3 x + \ldots \Big] c \quad . \end{aligned}$$

In the passage to the limit hold fixed the point x_α, but

allow the number of subdivisions to increase without limit while Δx approaches zero. If the subscript α is suppressed one obtains in the limit

$$y(x) = \left[I + \int_{x_0}^{x} M(\xi)d\xi + \int_{x_0}^{x} \int_{x_0}^{\xi} M(\xi)M(\xi')d\xi' d\xi \right.$$

$$\left. + \int_{x_0}^{x} \int_{x_0}^{\xi} \int_{x_0}^{\xi'} M(\xi)M(\xi')M(\xi'')d\xi'' d\xi' d\xi + \cdots \right] c \quad .$$

If the matrix

$$(6.1.2) \quad K(x) = I + \int_{x_0}^{x} M(\xi)d\xi + \int_{x_0}^{x} \int_{x_0}^{\xi} M(\xi)M(\xi')d\xi' d\xi + \cdots$$

is introduced, the solution becomes

$$(6.1.3) \qquad y(x) = K(x)c \quad ,$$

where $K(x)$ is a matrix whose columns constitute a fundamental set of solutions and which reduces to the identity at x_0.

The solution (6.1.3) can be obtained by a well-known classical procedure. First of all it is noted that the original equations

$$y' = M(x)y \quad , \qquad y(x_0) = c \quad ,$$

are equivalent to the set of equations

$$y(x) = c + \int_{x_0}^{x} M(\xi)y(\xi)d\xi \quad .$$

Thus, by repeated substitutions, one finds that

$$y(x) = c + \int_{x_0}^{x} M(\xi)\left[c + \int_{x_0}^{\xi} M(\xi')y(\xi')d\xi'\right] d\xi$$

$$= \left[I + \int_{x_0}^{x} M(\xi)d\xi\right] c + \int_{x_0}^{x}\int_{x_0}^{\xi} M(\xi)M(\xi')y(\xi')d\xi' d\xi$$

$$= \ldots$$

This procedure gives rise once again to the matrix K. It is known that this series converges to the solution, for an arbitrary c, under very general conditions. The general matrix K(x) of solutions, or the particular solution y(x), could be approximated by computing a sufficient number of terms in the series (6.1.2). The repeated integrations may, however, become very laborious or totally impossible to carry out in closed form. Numerical integration involves the breaking up of the range of integration into subintervals and the introduction of interpolation polynomials. It may be simpler and less laborious to apply the slab procedure for obtaining the approximation. Since the slab procedure is no more complicated when the slabs are unequal in thickness, it should be especially advantageous when certain parameters vary at highly non-uniform rates, permitting greater thicknesses where the rates are all low, and requiring smaller subdivisions where some are high.

It is worthy of particular note that this whole procedure

is quite independent of the form of M and can be applied to any set whatsoever of linear equations.

In the problem of determining the critical size of a pile, assumed to have a uniform interior region within a reflector of continuously varying parameters, if T is the total thickness of the reflector the foregoing procedure yields a relation of the form

$$y(a+T) = K(a+T)y(a) \quad . \tag{6.1.3}$$

In this relation the $\underline{a}$ appears explicitly in $y(a) = c$, but not at all in either $y(a+T)$ or $K(a+T)$, the $a+T$ being merely a place label, that is, $y(a+T)$ and $K(a+T)$ in (6.1.3) are functions of T alone. The $y(a+T)$ takes the place of $Y_\nu(t_\nu)c_\nu$ in § 2. Moreover,

$$y(a) = Y_0(a)c_0 = Y_0(a)y(0) \quad ,$$

if $Y_0(0) = I$. Accordingly, (5.1.3) can be written in the form

$$y(a+T) = K(a+T)Y_0(a)y(0) \quad ,$$

where $\underline{a}$ occurs in Y_0 alone. The first n components of $y(a+T)$ are zero and the last n components of $y(0)$ are zero. Thus it follows that in the 2n x 2n matrix $K(a+T)Y_0(a)$ the minor n x n matrix in the upper left-hand corner is singular. This is obtained by multiplying the upper half of K by the left hand

half of Y_0.

5.2. <u>The case of the spherical pile</u>. The discussion in the preceding section applies almost without change to the case of the spherical pile with a reflector with continuously varying parameters. There is, however, a significant difference in that the transformation employed above to obtain equations with coefficients that are constant within each spherical shell at the same time provides solutions which are discontinuous in their last n components at the boundaries. The substitution that provides continuity at the boundaries, viz. $y_{n+j} = \lambda_j y_j''$ yields a set of equations with coefficients which are functions of r even within each uniform shell.

The advantage of having equations with constant coefficients is, of course, that their solutions can be written down readily in terms of exponentials, or trigonometric and hyperbolic sines and cosines. However, once solutions are given in these terms for the equations with constant coefficients (with discontinuities in the solutions at the boundaries) a simple transformation provides the corresponding solutions of the equations with variable coefficients (with solutions that are continuous at the boundaries). A further readjustment may be needed to provide a matrix of solutions reducing to I at the inner boundary, but this can always be effected. From this point, the procedure is formally the same as for the slab case, and the spherical

shell approximation to the case of continuously varying parameters can be approximated by solving a finite number of sets of equations for uniform shells.

In conclusion it is noted that the form of the solution in the cylindrical case does not lend itself to the type of analysis used in the preceding sections for the problem of continuously varying parameters. At the present time then the cylindrical case with continuously varying parameters in the reflector constitutes an unsolved problem.

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